Assignment 3

This homework is due *Friday*, September 30.

There are total 30 points in this assignment. 26 points is considered 100%. If you go over 26 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations. Bare answers will not earn you much.

This assignment covers section 1.7, 2.1 in O'Neill.

(1) (1.7.3,4) Let F denote the mapping $F(u,v) = (u^2 - v^2, 2uv)$.
  (a) [2pt] Let $v = (v_1, v_2)$ be a tangent vector to $\mathbb{R}^2$ at $p = (p_1, p_2)$. Apply Definition of $F*$ directly to express $F*(v)$ through coordinates of v and p.
  (b) [2pt] Find a formula for the Jacobian matrix of F at all points. Find at which points the matrix is of rank 2.

(2) [2pt] (1.7.6a) Give an example to demonstrate that a bijective (one-to-one and onto) differentiable mapping need not be a diffeomorphism. (*Hint:* Take $m = n = 1$ and a function whose inverse is not differentiable.)

(3) (1.7.10) Show (in two ways) that the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(u,v) = (ve^u, 2u)$ is a diffeomorphism:
  (a) [2pt] Prove that it is one-to-one, onto, and regular.
  (b) [2pt] Find a formula for its inverse $F^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ and observe that F^{-1} is differentiable. Verify the formula by checking that both $F(F^{-1})$ and $F^{-1}(F)$ are identity maps.

(4) (2.1.1) Let $v = (1, 2, -1)$ and $w = (-1, 0, 3)$ be tangent vectors at a point of $\mathbb{R}^3$. Compute
  (a) [1pt] $v \cdot w$,
  (b) [1pt] $v \times w$,
  (c) [1pt] $v/\|v\|$, $w/\|w\|$,
  (d) [1pt] $\|v \times w\|$,
  (e) [1pt] the cosine of the angle between v and w.

(5) [2pt] (2.1.3) Prove that the tangent vectors
$$e_1 = \frac{(1,2,1)}{\sqrt{6}}, \quad e_2 = \frac{(-2,0,2)}{\sqrt{8}}, \quad e_3 = \frac{(1,-1,1)}{\sqrt{3}}$$
constitute a frame. Express $v = (6, 1, -1)$ as a linear combination of these vectors.

— *see next page* —

(6) [2pt] (2.1.7) If e is a unit vector, then the *component* of v in the e direction is

$$(v \cdot e)e = \|v\| \cos\theta \; e.$$

Show that v has a unique expression $v = v_1 + v_2$, where $v_1 \cdot v_2 = 0$ and v_1 is the component of v in the e direction.

(7) (a) [2pt] (2.1.5) Prove that $v \times w \neq 0$ if and only if v and w are linearly independent, and show that $\|v \times w\|$ is the area of the parallelogram with sides v and w.

(b) [2pt] (2.1.8)Prove that the volume of the parallelepiped with sides u, v, w is $|v \cdot (v \times w)|$. (*Hint:* Use the previous item and use $e = v \times w / \|v \times w\|$ to express $v \times w$.)

(8) (Based on 2.1.11) For a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$, denote

$$\nabla f = \sum \frac{\partial f}{\partial x_i} U_i = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$

(a) [2pt] Show that directional derivative of f is the dot product of direction and ∇f, i.e. $v[f] = v \cdot \nabla f|_p$.

(b) [2pt] Given a nonzero vector $a = (a_1, a_2, a_3)$, find a *unit* vector $e = (e_1, e_2, e_3)$ such that $a \cdot e \geq a \cdot e'$ for any *unit* vector e'. (In other words, find a unit vector s.t. $a \cdot e$ reaches its maximum over all unit vectors.)

(c) [1pt] Find a unit vector e such that $e[f]$ reaches its maximum over all unit vectors. (*Hint:* Use the previous two questions.)

(d) [2pt] Find the value of directional derivative $e[f]$ that you found in the previous item. (*Hint:* Answer is $\left(\sum (\partial f / \partial x_i)^2\right)^{1/2}$.)

Comment. ∇f is also denoted $\operatorname{grad} f$ and is called the gradient of f. This problem explains the statement "gradient is the direction of fastest growth of a function".